

Relativistic Hartree-Fock schemes and effect of self-consistency

S.S. Wu, J.M. Zhu, K.Z. Liu, Y.J. Yao

Center for Theoretical Physics and Department of Physics, Jilin University, Changchun 130023, P.R. China

Received: 12 March 1999 / Revised version: 6 September 1999

Communicated by P. Schuck

Abstract. A new effect of self-consistency in the relativistic Hartree-Fock (HF) approximation is studied by a simple model and a renormalized calculation. A comparison is made between two different HF schemes: one requiring self-consistency in the HF potential (scheme P) and the other in the baryon propagator (scheme BP). Our results show that scheme P is a good approximation to scheme BP for the calculation of the baryon propagator and the self-consistency requirements make the results obtained by the two schemes closer to each other, because the self-consistency in scheme BP diminishes the continuum part of the spectral representation for the baryon propagator, while the self-consistency in scheme P yields a baryon propagator which approximates closely to the HF result contributed by the converged single particle part of the above spectral representation alone.

PACS. 21.65.+f Nuclear matter – 21.60.Jz Hartree-Fock and random-phase approximations – 11.10.Gh Renormalization

Not only the relativistic Hartree-Fock (HF) [1] but also the relativistic Brueckner-HF [2] theory has been studied intensively in recent years. The aim of this note is somewhat different. We would like to compare two different relativistic self-consistent (SC) HF schemes and to study a new effect of self-consistency (i.e. besides its relation with the energy variational principle) by a renormalized calculation. For simplicity we shall consider a system composed of baryons coupling only with scalar mesons and the case of zero-density. The baryon tadpole self-energy is then zero and there is no distinction between the HF and the Fock approximation, though we shall still use the term HF. The baryon propagator can be written as

$$G_{HF}(k) = -[\gamma_\mu k_\mu - iM + \Sigma_{HF}^x(k)]^{-1} \quad (1a)$$

$$= G^0(k) + G^0(k)\Sigma_{HF}^x(k)G_{HF}(k),$$

$$G^0(k) = -[\gamma_\mu k_\mu - iM]^{-1}, \quad (1b)$$

where $k_\mu = (\mathbf{k}, k_4 = ik_0)$ and the renormalized HF exchange self-energy can be expressed in the form:

$$\Sigma_{HF}^x(k) = -g_s^2 \int \frac{d^\tau q}{(2\pi)^4} G(q)\Delta^0(k-q) + \Sigma_{CTC}^x(k) \quad (2)$$

with $\tau = 4 - \varepsilon$ ($\varepsilon \rightarrow 0+$) and CTC denoting the counterterm correction. Substituting $G_{HF}(q)$ for $G(q)$ in (2), one gets

$$\Sigma_{HF}^x(k) = g_s^2 \int \frac{d^\tau q}{(2\pi)^4} \cdot \frac{\Delta^0(k-q)}{\gamma_\mu k_\mu - iM + \Sigma_{HF}^x(q)} + \Sigma_{CTC}^x(k), \quad (3)$$

which is the SC equation considered by Bielajew and Serot [3] (see also [4]). It will be referred to as the semifully SC scheme or simply as scheme BP, since $\Delta^0(k-q) = -i[(k-q)^2 + m_s^2 - i\varepsilon]^{-1}$ is still the free meson propagator. From (1) one obtains the following eigenvalue equation

$$[\gamma_\mu k_\mu - iM + \Sigma_{HF}^x(k)]_{k_0=E_k} u(k_s) = 0, \quad (4)$$

where E_k and $u(k_s)$ are the eigenvalue and eigenspinor, respectively. Let the zero-order approximation to $G_{HF}(k)$ constructed by means of (4) be denoted by $G_\Sigma^0(k)$. We have

$$G_\Sigma^0(k) = -[\gamma_\mu k_\mu - iM + \Sigma_{HF}^x(\mathbf{k}, E_k)]^{-1}, \quad (5)$$

$$G_{HF}(k) = G_\Sigma^0(k) + G_\Sigma^0(k) [\Sigma_{HF}^x(k) - \Sigma_{HF}^x(\mathbf{k}, E_k)] G_{HF}(k), \quad (6)$$

where $\Sigma_{HF}^x(\mathbf{k}, E_k)$ represents $\Sigma_{HF}^x(k)$ at $k_0 = E_k$. It has been conjectured in the HF theory [5] that $G_\Sigma^0(k) \approx G_{HF}(k)$. Hence it may be a good approximation to (3) if one substitutes $G_\Sigma^0(q)$ for $G(q)$ in (2), from which one obtains

$$\Sigma_{HF}^x(k) = Z_t g_s^2 \int \frac{d^\tau q}{(2\pi)^4} \cdot \frac{\Delta^0(k-q)}{\gamma_\mu k_\mu - iM + \Sigma_{HF}^x(\mathbf{q}, E_q)} + \Sigma_{CTC}^x(k), \quad (7)$$

with $Z_t = 1$. It is referred to as the potential scheme [6] or scheme P, since if one sets $k_0 = E_k$ in (7), one gets a SC equation for $\Sigma_{HF}^x(\mathbf{k}, E_k)$. From the stipulation that $\Delta G(k) = G(k) - G_\Sigma^0(k)$ should be small and the single

particle (sp) potential instantaneous, i.e. independent of k_0 , one easily concludes that $G_\Sigma^0(k)$ and $G(k)$ should have the same real pole (C1) and the same residue ($-Z_t$) at the pole (C2). Clearly $G_\Sigma^0(k)$ satisfies (C1). Following the notation used in [7], we shall denote the pole of $G(k)$ by $\gamma_\mu k_\mu = iM_t$. $\Sigma^x(k)$ can be written as [3]

$$\Sigma^x(k) = \gamma_\mu k_\mu a(k^2) - iM_t b(k^2). \quad (8)$$

One easily finds that Z_t can be written in the form

$$Z_t = \left[1 + \frac{\partial \Sigma^x}{\partial (\gamma_\mu k_\mu)} (\gamma_\mu k_\mu = iM_t) \right]^{-1}, \quad (9a)$$

$$\frac{\partial \Sigma^x}{\partial (\gamma_\mu k_\mu)} (\gamma_\mu k_\mu = iM_t) = a(-M_t^2) - 2M_t^2 a'(-M_t^2) + 2MM_t b'(-M_t^2), \quad (9b)$$

where $f'(k^2) = \frac{df}{dk^2}$. From (4) one gets that the root M_t is determined by

$$M^2 [1 + b(-M_t^2)]^2 - M_t^2 [1 + a(-M_t^2)]^2 = 0, \quad (10)$$

and $E_k^2 = \mathbf{k}^2 + M_t^2$ [6]. If $Z_t \neq 1$, $Z_t G_\Sigma^0(k)$ should be a better approximation to $G_{HF}(k)$ than $G_\Sigma^0(k)$. Thus we have added the factor Z_t to (7). We shall use a tilde and a caret to indicate the results obtained from (3) and (7), respectively. Since in the case of zero-density (3) and (7) can be solved rigorously, it is advantageous to use this fact to make a preliminary study of the relation between different approximations and the effect of self-consistency.

To fix $\Sigma_{CTC}^x(k)$ we shall consider the intermediate renormalization conditions [7], which read

$$\Sigma^x(k) \Big|_{k_\mu=0} = 0; \quad \frac{\partial \Sigma^x(k)}{\partial (\gamma_\mu k_\mu)} \Big|_{k_\mu=0} = 0. \quad (11)$$

The solutions to (3) and (7) with $Z_t = 1$ have been given in [6]. According to [6], $G_\Sigma^0(k)$ can be written as

$$G_\Sigma^0(k) = -[\gamma_\mu k_\mu - iM_t]^{-1}, \quad (12)$$

where M_t may be interpreted as the true baryon mass and M is then determined by (10). As is wellknown [7], a properly normalized spectral representation for the baryon propagator can be written as

$$G_{HF}^n = -Z_2 \frac{\gamma_\mu k_\mu + iM_t}{k^2 + M_t^2 - i\varepsilon} - \int_{m_1^2}^{\infty} dm^2 \frac{\gamma_\mu k_\mu \alpha(-m^2, Z_2) + iM_t \beta(-m^2, Z_2)}{k^2 + m^2 - i\varepsilon}, \quad (13)$$

where $m_1 = M_t + m_s$ and Z_2 satisfies

$$Z_2 + \int_{m_1^2}^{\infty} dm^2 \alpha(-m^2, Z_2) = 1. \quad (14)$$

Between G_{HF}^n and G_{HF} in (1a) we should write $G_{HF}^n = ZG_{HF}$, Z being a proportional factor. Since the residues

at the pole $\gamma_\mu k_\mu = iM_t$ must be the same, we get $Z_2 = ZZ_t$ with Z_t given by (9). (1) implies that one should substitute $G_{HF} = Z^{-1}G_{HF}^n$ in (2). Let γ denote α or β . Set $\gamma(-m^2) = Z^{-1}\gamma(-m^2, Z_2)$. As shown in [6], inserting (8) and (13) in (2) and using (11), one finds in scheme BP

$$\begin{aligned} \tilde{a}(k^2) &= -\frac{g_s^2}{16\pi^2} \int_0^\infty dm^2 \int_0^1 dx f_\alpha(-m^2) \\ &\cdot x \ln \frac{K^2(x, m^2, k^2)}{K^2(x, m^2, 0)}, \end{aligned} \quad (15a)$$

$$\begin{aligned} \tilde{b}(k^2) &= \frac{g_s^2}{16\pi^2} \left[\frac{M_t}{M} \right] \int_0^\infty dm^2 \int_0^1 dx f_\beta(-m^2) \\ &\cdot \ln \frac{K^2(x, m^2, k^2)}{K^2(x, m^2, 0)}, \end{aligned} \quad (15b)$$

$$f_\gamma(-m^2) = \frac{\tilde{Z}_t \delta(m^2 - M_t^2)}{+\theta(m^2 - m_1^2) \tilde{\gamma}(-m^2)}, \quad (15c)$$

$$K^2(x, m^2, k^2) = (1-x)m^2 + xm_s^2 + x(1-x)k^2, \quad (15d)$$

where θ denotes the step function. By means of $G_{HF} = Z^{-1}G_{HF}^n$ and the fact that the spectral weight functions α and β should be real, one can derive the following relations [8]:

$$\alpha(k^2) = \frac{1}{\pi} Im \frac{1 + a(k^2)}{D(k^2)}, \quad (16a)$$

$$\beta(k^2) = \frac{M}{\pi M_t} Im \frac{1 + b(k^2)}{D(k^2)}, \quad (16b)$$

$$D(k^2) = k^2 (1 + a(k^2))^2 + M^2 (1 + b(k^2))^2. \quad (16c)$$

It is clear that (16) applies to both schemes BP and P. We note that Z_2 in (18c) of [6] should read $Z_t = Z^{-1}Z_2$. In scheme BP the set of equations (15), (16), (9) and (10) must be solved self-consistently. They have been solved by iteration. If the on-shell renormalization conditions are used, Z_t is equal to 1. Thus it is in this aspect simpler. Bielajew [8] has calculated the spectral weight functions of this case numerically. For comparison we shall use his parameter values $\frac{g_s^2}{16\pi^2} \equiv \bar{g}_s^2 = 0.6517$, $\frac{m_s}{M_t} = 0.5538$ and $M_t = 4.7585 fm^{-1}$. The numerical results of (\tilde{a}, \tilde{b}) and $(\tilde{\alpha}, \tilde{\beta})$ are drawn in Figs. 1 and 2. In Fig. 1 the subscript $r(i)$ indicates the real (imaginary) part. Note that $Z_2 = ZZ_t$ can be calculated by (14). The values of \tilde{Z}_t , \tilde{Z}_2 and $(\frac{M_t}{M})$ will be given below. Substituting (12) in (7) and employing (11), we obtain in scheme P [6]

$$\hat{a}(k^2) = \hat{Z}_t A(k^2); \quad \hat{b}(k^2) = \hat{Z}_t \left(\frac{M_t}{M} \right) B(k^2), \quad (17a)$$

$$A(k^2) = -\bar{g}_s^2 \int_0^1 dx [x \ln L(x, k^2)]; \quad (17b)$$

$$B(k^2) = \bar{g}_s^2 \int_0^1 dx \ln L(x, k^2),$$

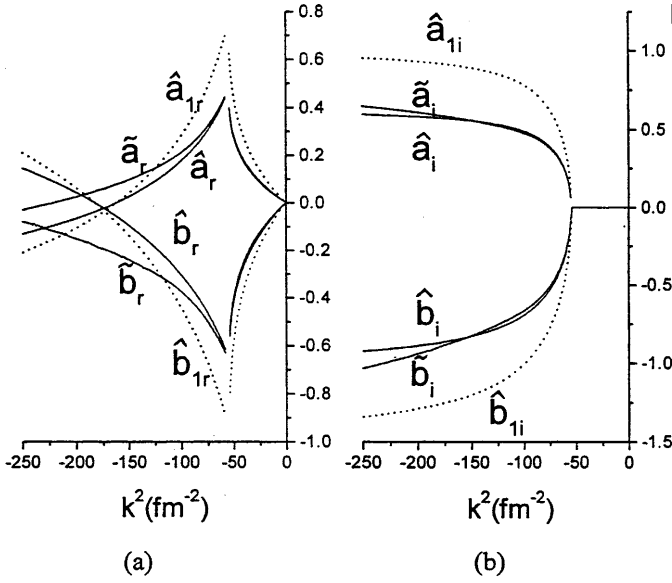


Fig. 1. Numerical results of (a,b). (a) the real part, (b) the imaginary part. The tilde indicates scheme BP, while the caret scheme P. The dotted curves are obtained by taking $Z_t = 1$ in (7)

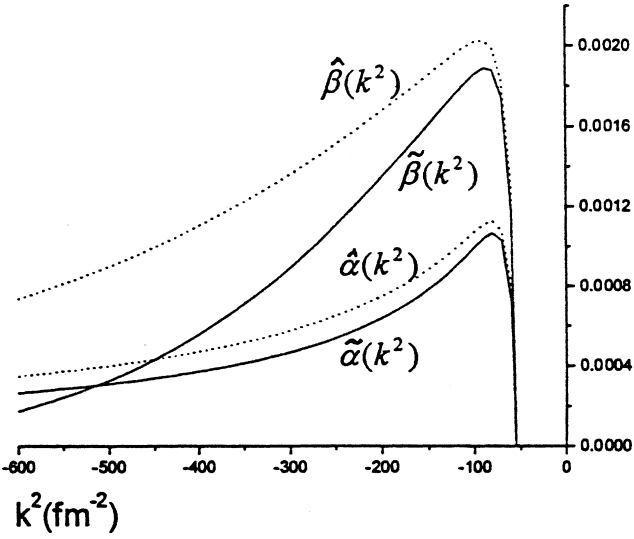


Fig. 2. The spectral weight functions $\alpha(-m^2)$ and $\beta(-m^2)$. The tilde for scheme BP, while the caret for scheme P

$$L(x, k^2) = \left[(1-x)M_t^2 + xm_s^2 + x(1-x)k^2 \right] \cdot \left[(1-x)M_t^2 + xm_s^2 \right]^{-1} \quad (17c)$$

where \widehat{Z}_t and \widehat{M} are given by (9) and (10), respectively. Both \widehat{Z}_t and \widehat{M} depend on \widehat{a} and \widehat{b} . However, the dependence occurs only at a single point $k^2 = -M_t^2$. Thus (17a) can be solved easily without even using an iteration procedure. We note that $A(k^2)$ and $B(k^2)$ are known functions of k^2 , because M_t denotes the true baryon mass. Thus the righthand sides (RHSs) of (17a) will be known, if $\widehat{a}(-M_t^2)$ and $\widehat{b}(-M_t^2)$ are known. Setting $k^2 = -M_t^2$ in (17a) and

using (9) and (10), we obtain

$$\widehat{a}(-M_t^2) = 2^{-1} \left(-Y + [Y^2 + 4YA(-M_t^2)]^{\frac{1}{2}} \right), \quad (18a)$$

$$Y = A(-M_t^2) \left\{ A(-M_t^2) + 2M_t^2 \cdot [B'(-M_t^2) - A'(-M_t^2)] \right\}^{-1}, \quad (18b)$$

$$\widehat{b}(-M_t^2) = \widehat{a}(-M_t^2) B(-M_t^2) \left\{ A(-M_t^2) + \widehat{a}(-M_t^2) \cdot [A(-M_t^2) - B(-M_t^2)] \right\}^{-1}. \quad (18c)$$

The above demonstrates that scheme P is much simpler than scheme BP. If we have set $Z_t = 1$ in (7) and (17a), then $\widehat{a}(k^2) = A(k^2)$ is known and (18c) reduces to

$$\widehat{b}(-M_t^2) = B(-M_t^2) [1 + A(-M_t^2) - B(-M_t^2)]^{-1}. \quad (19)$$

We shall use an additional subscript 1 to indicate this case and refer to it as Cal. For comparison $(\widehat{a}, \widehat{b})$ and $(\widehat{a}_1, \widehat{b}_1)$ are also plotted in Fig. 1. The converged values of Z_t and $\frac{M_t}{M}$ are as follows: $\widetilde{Z}_t = 0.6083$, $\widehat{Z}_t = 0.6248$ and $\widehat{Z}_{1t} = 0.5099$, while $\frac{M_t}{M} = 0.8131$, $\frac{M_t}{M} = 0.8261$ and $\frac{M_t}{M_1} = 0.7048$. It is seen that $\widehat{a}_1(\widehat{b}_1)$ differs widely from $\widetilde{a}(\widetilde{b})$ and $\widehat{Z}_{1t}(\frac{M_t}{M_1})$ is also a poorer approximation to $\widetilde{Z}_t(\frac{M_t}{M})$ than $\widehat{Z}_t(\frac{M_t}{M})$. The difference between Cal and (17a) lies only in the fact that in Cal Z_t is not included in the SC requirement. The above results show that it is important to substitute $Z_t G_\Sigma^0(k)$ for $G_\Sigma^0(k)$ if Z_t is not close to 1 and the self-consistency makes scheme P and scheme BP closer to each other. We shall no longer consider Cal. The region $k^2 > (<) -m_1^2$ will be designated by I (II). In region I $(\widetilde{a}, \widetilde{b})$ and $(\widehat{a}, \widehat{b})$ are real and they are almost the same. They become complex in region II and are still close to each other for $k^2 > -120 (fm^{-2})$. However, their difference becomes larger and larger hereafter. To study the significance of this behavior, let us consider

$$F(\pm) \equiv \langle iG(k) \rangle_{\pm} = u_{\pm}^{\dagger}(ks) iG(k) u_{\pm}(ks), \quad (20)$$

where $+$ ($-$) refers to the eigenspinor of (4) with eigenvalue $+$ ($-$) $E_k = [M_t^2 + k^2]^{\frac{1}{2}}$. From (1), (4) and (8) we find

$$F(\pm) = \frac{1 + a(k^2)}{D(k^2)} \left[\frac{1 + b(k^2)}{1 + a(k^2)} M \pm \frac{M_t}{E_k} k_0 \right]. \quad (21)$$

For simplicity only $F_0 = \langle iG_\Sigma^0(k) \rangle_+$, $\widetilde{F}_r = Re \langle i\widetilde{G}_{HF}(k) \rangle_+$ and $\widehat{F}_r = Re \langle i\widehat{G}_{HF}(k) \rangle_+$ are depicted in Fig. 3, where $k_0 = 10\sqrt{2}(fm^{-1})$ is chosen as an example. One observes that \widehat{F}_r and \widetilde{F}_r are very close to each other. It shows that the difference between $(\widehat{a}, \widehat{b})$ and $(\widetilde{a}, \widetilde{b})$ at larger $|k^2|$ will not cause significant effects

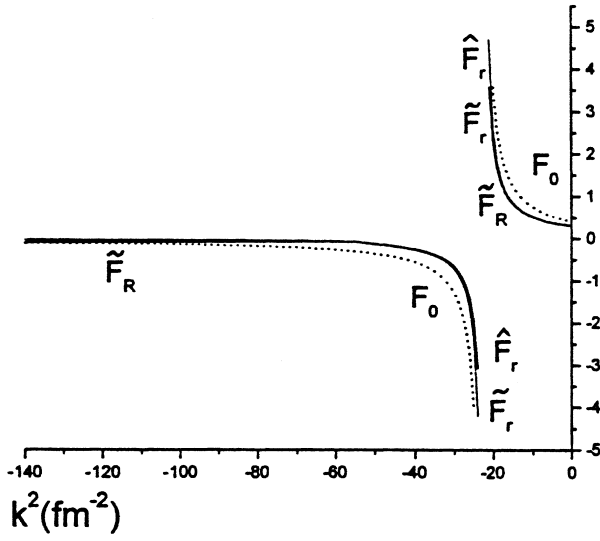


Fig. 3. Graphical representations for $F(+)=\langle iG(k) \rangle_+$, where $F_0 = \langle iG_{\Sigma}^0(k) \rangle_+$ (dotted curves), $\tilde{F}_r = \text{Re}\langle i\tilde{G}_{HF}(k) \rangle_+$, $\hat{F}_r = \text{Re}\langle i\hat{G}_{HF}(k) \rangle_+$ and $\tilde{F}_R = \text{Re}\langle i\tilde{G}_{HF}(\mathbf{k}, R) \rangle_+$. Without greatly enlarging the scale, \tilde{F}_r , \hat{F}_r and \tilde{F}_R are almost indistinguishable

on G_{HF} , because both \hat{G}_{HF} and \tilde{G}_{HF} become very small there. Hence, for the calculation of G_{HF} one may regard (\hat{a}, \hat{b}) as a good substitute for (\tilde{a}, \tilde{b}) . In Fig. 3 we have also plotted $\tilde{F}_R = \langle i\tilde{G}_{HF}(k; R) \rangle_+$ obtained in the quasiparticle approximation [9], where R indicates that a_i and b_i are neglected. From (21) we have

$$F_r = \text{Re}F = [D_r^2 + D_i^2]^{-1} \{D_r [M(1+b_r) + E_k^{-1}M_t k_0(1+a_r)] + D_i [M b_i + E_k^{-1}M_t k_0 a_i]\}, \quad (22a)$$

$$F_R = D_R^{-1} \{M(1+b_r) + E_k^{-1}M_t k_0(1+a_r)\}, \quad (22b)$$

$$D_r = \text{Re}D = D_R - (k^2 a_i^2 + M^2 b_i^2), \quad (23a)$$

$$D_R = k^2(1+a_r)^2 + M^2(1+b_r)^2, \quad (23b)$$

$$D_i = \text{Im}D = k^2 2a_i(1+a_r) + M^2 2b_i(1+b_r). \quad (23c)$$

Clearly F_R and F_r as well as D_R and D are the same in region I, because there $a_i = b_i = 0$. In region II the root structure of D_R and D may be different. Let M_R denote the smallest real root of D_R which does not belong to D . From (22) it is seen that F_R will deviate from F_r widely if $k^2 \lesssim -M_R^2$. Thus M_R may serve as a crude criterion in this aspect. In our special example we have found that both D_R and D have no real roots in region II and \tilde{F}_R and \tilde{F}_r are very close to each other, i.e. the quasiparticle approximation is quite good in this case. We have also calculated the case $k_0 = 10(\text{fm}^{-1})$. The relations obtained for $F_0, \hat{F}_r, \tilde{F}_r$ and \tilde{F}_R are nearly the same as shown in Fig.3. By means of (16) we may calculate $(\hat{\alpha}, \hat{\beta})$. The numerical results are shown in Fig.2. Since

$\hat{Z}_t \simeq \tilde{Z}_t$ and $\frac{M_t}{M} \simeq \frac{M_t}{M}$, (17a) is almost the same as the sp part in (15) obtained by setting $\tilde{\alpha}(-m^2) = \tilde{\beta}(-m^2) = 0$. The set (α, β) calculated with (a, b) obtained by this sp part has not been depicted in Fig.2, because it almost coincides with $(\hat{\alpha}, \hat{\beta})$ and may well be represented by the latter. Though both $\tilde{\beta}$ and $\hat{\beta}$ are positive, we have checked numerically that for each set (α, β) the relation $m\alpha(-m^2) - M_t\beta(-m^2) \geq 0$ [7] holds well for $m \geq m_1$. Let us choose $\tilde{Z}_t, \frac{M_t}{M}$ and $\tilde{\alpha}(-m^2) = \tilde{\beta}(-m^2) = 0$ as the initial input and solve the set of SC equations in scheme BP iteratively. From Fig.2 one observes that the iteration process changes (α, β) from $(\hat{\alpha}, \hat{\beta})$ to $(\tilde{\alpha}, \tilde{\beta})$, while keeps the sp part unchanged, because we have used the converged values of $Z_t, \frac{M_t}{M}$ as our input. This asserts that the effect of self-consistency is to diminish the continuum part of the spectral representation in (13) and make its sp part more important. Using (14) and the results:

$$\int_{m_1^2}^{\infty} dm^2 \hat{a}(-m^2) = 0.6128; \quad (24)$$

$$\int_{m_1^2}^{\infty} dm^2 \tilde{a}(-m^2) = 0.4643,$$

we get $\hat{Z}_2 = 0.5048$ and $\tilde{Z}_2 = 0.5671$. It shows that Z_2 is indeed increased through diminishing $\alpha(-m^2)$. Fig.9 in [8] indicates that the difference between $(\hat{\alpha}, \hat{\beta})$ and $(\tilde{\alpha}, \tilde{\beta})$ is larger in the case of the on-shell renormalization. Thus the above effect will be more distinct in this case, which will be discussed in some detail elsewhere. From Fig.2 one notes that $\tilde{\alpha}(-m^2)$ and $\tilde{\beta}(-m^2)$ are very small. This is also one of the reasons why the contribution of the continuum part is insignificant and scheme P is a good approximation to scheme BP for the calculation of the baryon propagator. However we must remark that for the calculation of (α, β) scheme P may not be regarded as an equally good approximation to scheme BP, especially if the effect is accumulative, as can be seen from (24) and Fig.2. In this note we have only discussed a simple and special case. The question whether the conclusions reached are valid generally has still to be studied. A fuller report and a more detailed discussion of our findings for the $\sigma - \omega$ model will be presented in a succeeding paper.

This work is supported in part by the National Natural Science Foundation of China, the Foundation of Education Ministry and Center of Theoretical Nuclear Physics, National Laboratory of Heavy Ion Accelerator in Lanzhou.

References

1. Serot, B.D., Walecka, J.D.: Adv. Nucl. Phys. **16**, 1 (1986)
2. Anastasio, M.R., Celenza, L.S., Pong, W.S., Shakin, C.M.: Phys. Rep. **100**, 327 (1983); Brockmann, R., Machleidt,

- R.: Phys. Lett. **149B**, 283 (1984); Phys. Rev. **C42**, 1965 (1990); Horowitz, C.J., Serot, B.D.: Nucl. Phys. **A464**, 613 (1987); Haar, B.t., Malfliet, R.: Phys. Rep. **149**, 207 (1987); Muether, H., Machleidt, R., Brockmann, R.: Phys. Rev. **C42**, 1981 (1990); Serot, B.D.: Rep. Prog. Phys. **55**, 1855 (1992); Sehn, L., Fuchs, C., Faessler, A.: Phys. Rev. **C56**, 216 (1997) and references quoted in the last two references
3. Bielajew, A.F., Serot B.D.: Ann. Phys. **156**, 215 (1984)
 4. Fetter, A.L., Walecka, J.D.: Quantum Theory of Many-Particle Systems. New York: McGraw-Hill 1971
 5. Jones, R.W., Mohling, F., Becker, R.L.: Nucl. Phys. **A220**, 45 (1974)
 6. Wu, S.S., Yao, Y.J.: Eur. Phys. J. **A3**, 49 (1998)
 7. Bjorken, J.D., Drell, S.D.: Relativistic Quantum Fields. New York: McGraw-Hill 1965
 8. Bielajew, A.F.: Nucl. Phys. **A404**, 428 (1983)
 9. Botermans, W., Malfliet, R.: Phys. Rep. **198**, 115 (1990)